Math 53, Discussions 116 and 118 **2022-10-10**

Local maxima and minima

Questions

Question 1. Suppose that f is a function such that $\nabla f(2,4) = \langle 0,0 \rangle$ and

$$\begin{bmatrix} f_{xx}(2,4) & f_{xy}(2,4) \\ f_{xy}(2,4) & f_{yy}(2,4) \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -2 \end{bmatrix}.$$

Is $(2,4)$ a local minimum, local maximum, saddle point, or is there not enough information to tell?

Question 2. Let $f(x,y)$ be a (nice) function whose domain is the entirety of $\mathbb{R}^2$. Which of the following statements are true?

(a) f can have a local maximum but no absolute maximum.
(b) f can have an absolute maximum but no local maximum.
(c) f must have an absolute maximum when constrained to $xy = 1$.
(d) If f has a local minimum or maximum at a point, then $\nabla f = \mathbf{0}$ at that point.
(e) If $\nabla f = \mathbf{0}$ at a point, then f has a local minimum or maximum at that point.
(f) f must have an absolute maximum when constrained to $x^2 + y^2 \leq 1$.

Question 3 (Stewart §14.7 #40)**.** Consider the function $f(x,y) = 3xe^y - x^3 - e^{3y}$.

(a) f has exactly one critical point. Find it.
(b) Using the 2nd derivative test, show that this critical point is a local maximum.
(c) Is this point also an absolute maximum?

Below are brief answers to the worksheet exercises. If you would like a more detailed solution, feel free to ask me in person. (Do let me know if you catch any mistakes!)

Answers to questions

Question 1. The determinant of the given matrix is 2, which is positive. Since $f_{xx}(2,4) < 0$, we conclude from the second derivative test that f has a local maximum at $(2,4)$.

Question 2.

(a) True. $f(x,y) = 3x - x^3$ has local maxima at all points $(1, y)$, but no absolute maxima.
(b) False. An absolute maximum is automatically also a local maximum.
(c) False. The function $f(x,y) = x$ has no maximum on this constraint curve.
(d) True. (Note that we assume $f(x,y)$ is nice and differentiable in the problem.)
(e) False. It could have a saddle point.
(f) True. This is a consequence of the Extreme Value Theorem, because $x^2 + y^2 \le 1$ is a closed and bounded region.

Question 3.

(a) We need to solve the system of equations

$$f_x(x,y) = 3e^y - 3x^2 = 0$$
$$f_y(x,y) = 3xe^y - 3e^{3y} = 0.$$

The second gives $x = e^{2y}$, which we can plug into the first to obtain $e^y - e^{4y} = 0$, or $e^{3y} = 1$. So $y = 0$, and $x = 1$.

(b) The matrix of second order partials is

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & 3e^y \\ 3e^y & 3xe^y - 9e^{3y} \end{bmatrix}$$

which at $(1,0)$ is

$$\begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}.$$

The determinant is 27, which is positive. Since $-6 < 0$, we conclude that $(1,0)$ is a local maximum.

(c) However, $(1,0)$ is not an absolute maximum. In fact, this function has no absolute maximum. You can see this already by looking at its behavior along the x-axis:

$$f(x,0) = 3x - x^3 - 1.$$

As $x \to -\infty$, this goes to ∞.